A RELATIONSHIP FOR PLASMA SHEATHS ABOUT LANGMUIR PROBES

by

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Langmuir probes have been in extensive use for a number of years in a variety of plasma studies. Of particular interest to the authors have been vehicle-ionospheric interactions. Interpretation of results, in most practical cases, has involved use of the Mott-Smith Langmuir (MSL) equations[1] which were derived some time ago. The MSL derivation avoided the electrostatic problem by assuming a uniform shielding distance ("Space Sheath") of arbitrary dimension and then solving the geometric problem. Use of these equations has usually involved estimating sheath dimensions in some indirect manner and then substituting this into the MSL equations to obtain ion and electron concentrations, temperatures, etc. This approach has obvious limitations, since the properties of the sheath are not fixed but are highly variable, i.e. at zero probe potential the sheath must vanish.

Walker[2] has treated, with some sophistication, the case of a spherical body in both monoenergetic and Maxwellian plasmas. The conservation of energy and angular momentum relations are used to obtain an equation for the ion and electron densities in terms of the local electric field. These equations used in conjunction with the Poisson equation are solved numerically. In addition to the electric field and the particle density distributions, Walker obtains the voltage-current probe characteristic curves. The calculations are, however, lengthy so that results must be displayed graphically or expressed by an approximate analytic expression. It is appealing to attempt to extract from these numerically calculated curves a relationship for the sheath which can be applied to the conventional MSL equations.

The MSL equation for a spherical body, in a Maxwellian plasma, and with an attractive potential on the probe, may be written as (Ref. 1, pg. 740):

$$\beta = X(1 - e^{-\psi/X}) \qquad \text{where:} \quad \beta = \frac{\tau}{\rho^2} - 1 \qquad (1)$$

$\tau = \frac{i}{n_o e h^2} \sqrt{\frac{m}{8\pi kT}}$

T = ion temp (^oK)

m = ion mass

k = Boltzmann constant

e = electronic charge

n_o = ambient charge density

i = current to the probe

h = Debye length = $\sqrt{\frac{kT}{4\pi e^2 n_o}}$

$\rho = \frac{r}{h}$

r = probe radius

$X = (\frac{s}{\rho} + 1)^2 - 1$

s = sheath thickness in Debye lengths

Expanding the exponential term and then allowing the sheath to become very large yields a limit of β for very large sheaths given by

$$\beta = \psi - \frac{1}{2}\frac{\psi^2}{X} + \frac{1}{6}\frac{\psi^3}{X^2}$$

$$\beta_{X \to \infty} = \psi \qquad (2)$$

Langmuir probe behavior is conventionally divided into two regimes, characterized by the relative dimensions of the radius and sheath. If the sheath is very large compared to the radius, i.e. in the region where Eq. (2) is applicable, the system is said to be orbit limited and the MSL equation is independent of s. Thus, if $\rho \ll s$ the results obtained are independent of any other assumptions concerning the sheath, and the given equations are accurate in their present form. At the other extreme the sheath is of the order of the radius or smaller, in which case, the probe is said to be operating sheath limited. Unfortunately, this is the situation which exists in most physical experiments.

Figure 1 is a plot of Walker's voltage-current characteristic curves and as may be seen, they are quite systematic. By taking ratios between curves at different values of ψ and applying a fixed correction term, one can obtain a function representing all points with $\rho > 1$, to an excellent approximation (see Fig. (2)). The slope of this line is in the vicinity of 0.75, suggesting an exponent of ψ near that value. However, we have an alternative means of estimating this exponent.

The fact that the sheath must vanish at zero potential, places a restriction on the functional relation between s and ψ. We can apply one other criterion. We expect that s would monotonically increase with ψ and that at sufficiently high potentials the sheath dimensions will become large compared to the radius and orbit limited behavior will hold forth, therefore,

$$\beta_{\psi \rightarrow \infty} \alpha \psi \tag{3}$$

as s becomes very large X approaches the value

$$X = \left(\frac{s}{\rho}\right)^2 \qquad \frac{s}{\rho} \gg 1 \tag{4}$$

If $\frac{\psi}{X} \rightarrow \infty$ as $\psi \rightarrow \infty$ that is if:

$$\frac{\ln s}{\ln \psi} < 0.5 \qquad (5)$$

Then Eq. (1) reduces to

$$\beta = X \quad \psi \to \infty \qquad (6)$$

Substituting Eq. (4) into (6) and equating to (3) we obtain

$$s^2 \alpha \psi \qquad (7)$$

It will be noted that (6) is not satisfied under these conditions but for this particular solution the results are unaffected. We need merely replace (5) with

$$\beta = AX \qquad \text{where:} \quad A = f(\rho) \qquad (8)$$

In order to obtain the ρ dependence we make use of the Walker results. After some fitting, the following empirical relation was obtained.

$$s = 0.83\psi^{\frac{1}{2}} \rho^{\frac{1}{3}} \qquad (9)$$

Fig. (3) is a plot using this relation. Figure (4) is an error plot showing the deviation using the MSL equation with (9) vs. Walker's calculations.

While the general agreement is rather good, considering our present state, it is apparent that a systematic error with ψ exists and we can, obviously, do better. The following function was found to give a somewhat better fit to the available data.

$$s = 0.67\psi^{\frac{1}{2}} \rho^{\frac{1}{3}} + 0.3\rho \, e^{-0.01/\psi} \qquad (10)$$

The error plot for this function is available in Fig. (5).

These general results may also be obtained by an approximation technique which consists of a calculation of the charge on a body. We then equate this to the charge enclosed by its associated sheath of thickness s. This relation may then be solved for s.

In the region where the MSL equation depends significantly on the sheath thickness, s, the charge density in the sheath[3] is approximately inversely proportional to the local velocity of ions and electrons*

$$n = n_0\, u/v \qquad \text{where: } u = \text{velocity outside sheath}$$
$$v = \text{local velocity} \tag{11}$$

By conservation of energy considerations,

$$\frac{1}{2}\, mv^2 = \frac{1}{2}\, mu^2 + e\, |V| \tag{12}$$

and substitute Eq. (12) into Eq. (11) we obtain for the density withing the sheath

$$n = n_0 \sqrt{1 + \frac{2}{3}\, \psi} \tag{13}$$

where $\psi = e\, |V|/kT$ and where we express $mu^2/2$ in terms of the thermal energy of the ambient plasma $\frac{1}{2}\, mu^2 = \frac{3}{2}\, kT$.

Let us now calculate the charge on the body in terms of ψ. Using for the capacitance the expression for two concentric spheres,

$$C_s = h\rho\, (\rho + s)/s \tag{14}$$

The charge on the body will be

$$Q_s = C_s V = kTh\, \psi_s\, \rho(\rho + s)/es \tag{15}$$

where e is the charge on the ion. The total charge in the sheath Q_s' is,

$$Q_s' = ne\, V_{ol} = \frac{4\pi}{3}\, eh^3\, [(\rho + s)^3 - \rho^3]\, n_0\, (1 + \frac{2}{3}\, \psi)^{-\frac{1}{2}} \tag{16}$$

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*Equation (11) is an approximation which does not hold for small values of ρ. A detailed consideration of this is given by E. J. Öpik.[3]

Equating Q_s and Q_s' we obtain

$$s^4 + 3s^3\rho + 3s^2\rho^2 = 3\rho\,(\rho + s)\,\psi_s\,(1 + \tfrac{2}{3}\,\psi_s)^{\frac{1}{2}} \tag{17}$$

If we assume the sheath to be thick compared with the radius of the body, $s \gg \rho$, then Eq. (17) becomes

$$s^3 = 3\rho\psi_s\,(1 + \tfrac{2}{3}\,\psi_s)^{\frac{1}{2}} \tag{18}$$

For large values of ψ_s Eq. (18) gives for s,

$$s = 6^{\frac{1}{6}}\,\rho^{\frac{1}{3}}\,\psi_s^{\frac{1}{2}} \tag{19}$$

If $\rho \gg s$, Eq. (17) becomes

$$s^2 = \psi_s\,(1 + \tfrac{2}{3}\,\psi_s)^{\frac{1}{2}} \tag{20}$$

and for large values of ψ_s Eq. (20) becomes

$$s = (\tfrac{2}{3})^{\frac{1}{4}}\,\psi_s^{\frac{3}{4}} \tag{21}$$

Except for the coefficients, Eqs. (9) and (19) are the same. These equations show that the sheath may be quite large compared with estimates of the sheath thickness using the Debye length as the scaling length.

Since the above analysis gave us the correct functional relationships, it is of some interest to extend the analysis to cylindrical geometry where the more rigorous calculations are not available. The charge on a unit length of a cylinder is:

$$Q_c = \frac{kT\,\psi_s}{2e\,\ln\,(1 + s/\rho)} \tag{22}$$

The charge in the sheath is

$$Q_s = 2en_oh^2\,(s^2 + 2\rho s)\,(1 + \tfrac{2}{3}\,\psi_s)^{-\frac{1}{2}} \tag{23}$$

Equating the two we obtain,

$$(s^2 + 2\rho s)\ \ln(1 + \frac{s}{\rho}) = \pi\psi_s \sqrt{1 + \frac{2}{3}\psi_s} \tag{24}$$

If $\rho \gg s$ Eq. (24) becomes

$$s = (\frac{\pi}{2}\psi_s \sqrt{1 + \frac{2}{3}\psi_s}\ )^{\frac{1}{2}} \tag{25}$$

and for large values of ψ_s,

$$s = 1.66\ \psi_s^{\frac{3}{4}} \tag{26}$$

Solving Eq. (24) for s where $s \gg \rho$, we obtain

$$s = [\pi\psi_s \sqrt{1 + \frac{2}{3}\psi_s}\ /\ln(\frac{s}{\rho})]^{\frac{1}{2}} \tag{27}$$

For large values of ψ_s this reduces to

$$s = [\frac{\pi\ \sqrt{2/3}}{\ln(s/\rho)}]^{\frac{1}{2}}\ \psi_s^{\frac{3}{4}} \tag{28}$$

Equations (26) and (28) are quite similar since the dependence of s on $(\ln s/\rho)^{\frac{1}{2}}$ is small.

Conclusions

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An analytical relation for the sheath about a spherical Langmuir probe has been empirically derived from calculated data which reproduces this data within a mean error of better than 10 per cent. A semi-quantitative analysis verifies the functional dependence of this relation and a similar analysis develops the corresponding relationship for cylindrical geometry. In the absence of sophisticated calculations in the cylindrical case a detailed evaluation of coefficients is not possible. The relations herein derived, even in their present relatively crude form, should considerably aid in analysis of experimental Langmuir probe results;

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in particular, that involving spherical geometry. Author

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We would like to acknowledge the encouragement of Prof. S. F. Singer in this work.

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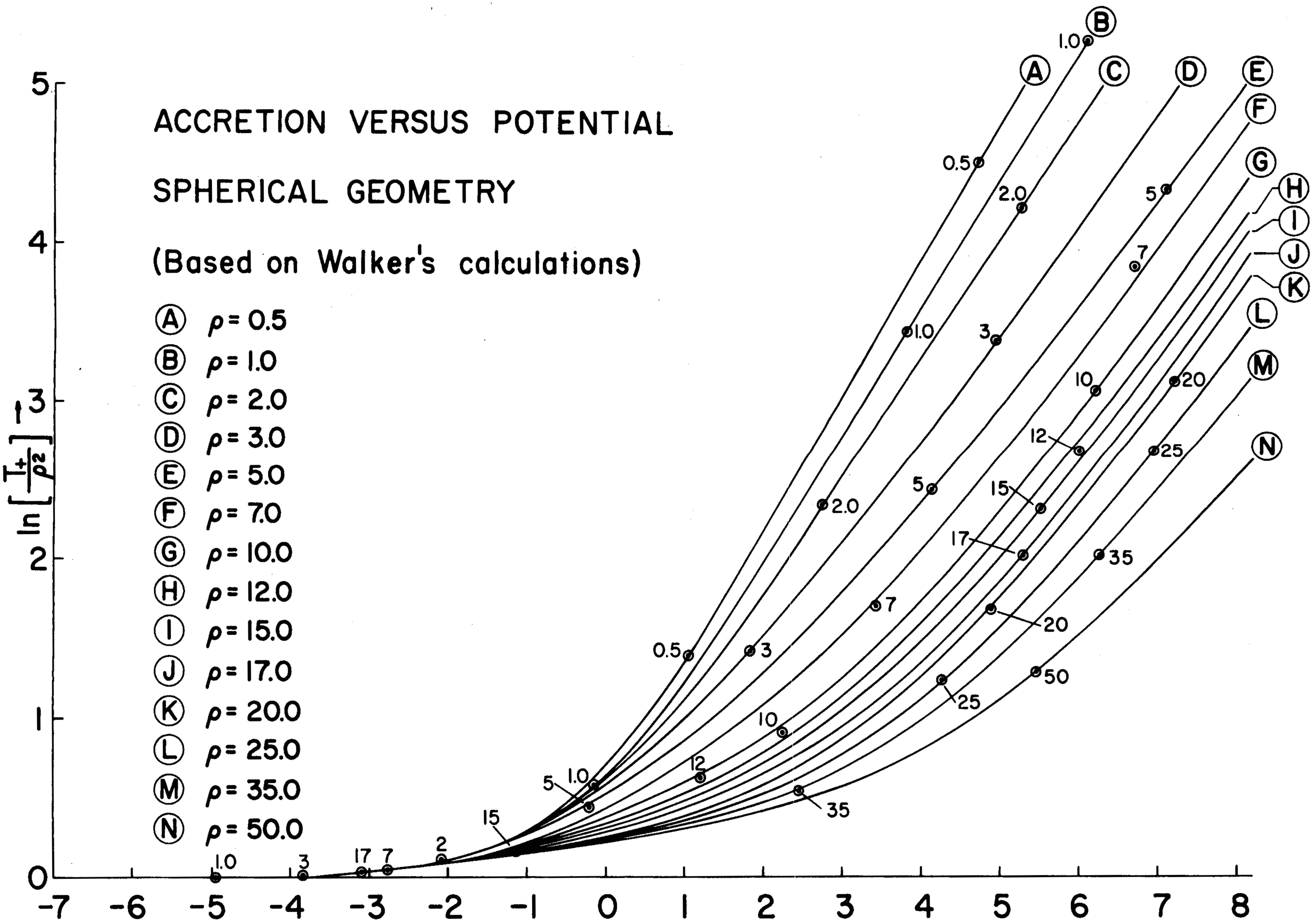
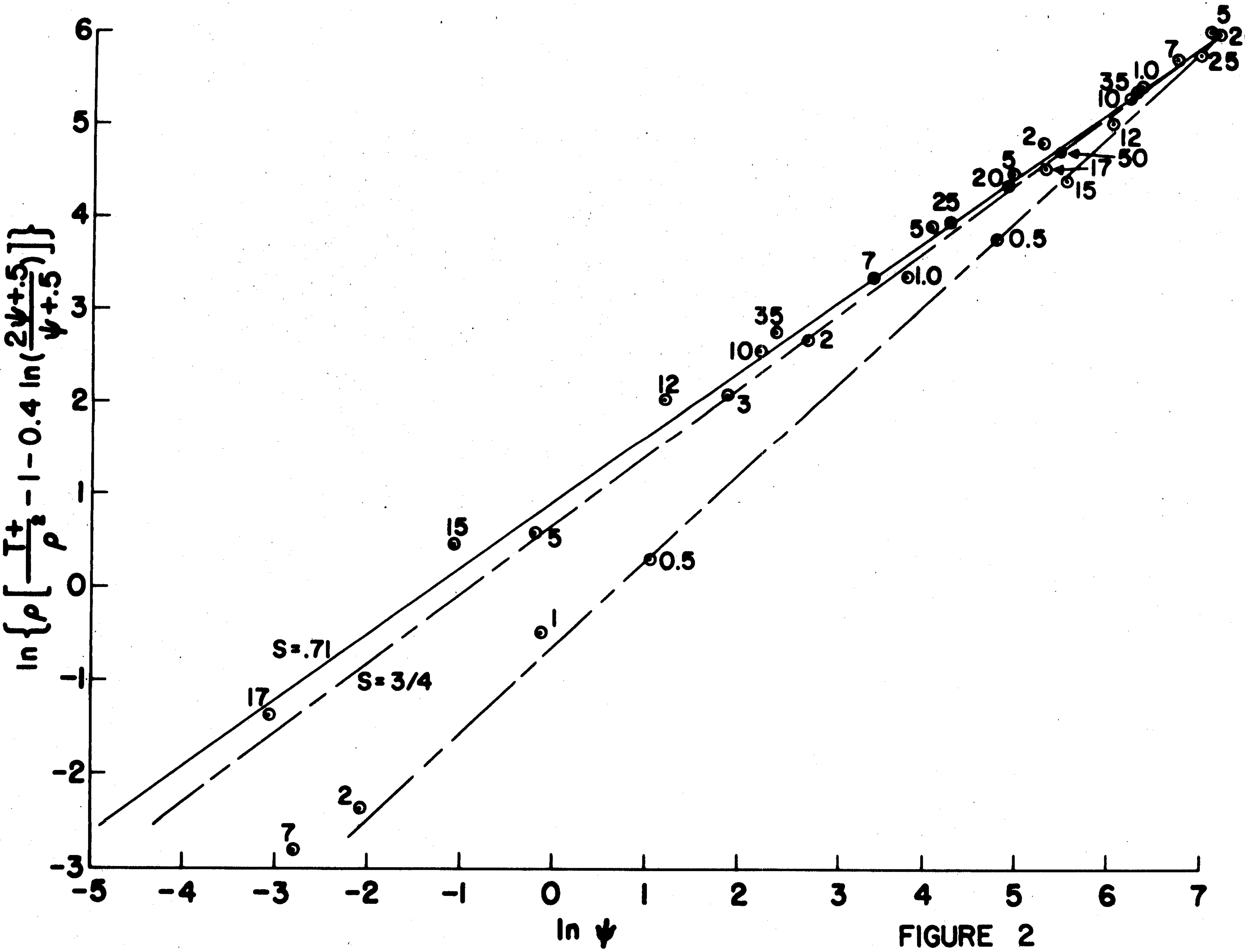


ACCRETION VERSUS POTENTIAL
SPHERICAL GEOMETRY
(Based on Walker's calculations)
Ⓐ $\rho = 0.5$
Ⓑ $\rho = 1.0$
Ⓒ $\rho = 2.0$
Ⓓ $\rho = 3.0$
Ⓔ $\rho = 5.0$
Ⓕ $\rho = 7.0$
Ⓖ $\rho = 10.0$
Ⓗ $\rho = 12.0$
Ⓘ $\rho = 15.0$
Ⓙ $\rho = 17.0$
Ⓚ $\rho = 20.0$
Ⓛ $\rho = 25.0$
Ⓜ $\rho = 35.0$
Ⓝ $\rho = 50.0$
$\ln\left[\frac{T_+}{\rho^2}\right] \rightarrow$
$\ln \psi$

FIGURE I



$\ln\left\{\rho\left[\frac{T+}{\rho^2}-1-0.4\ln\left(\frac{2\psi+.5}{\psi+.5}\right)\right]\right\}$
$\ln \psi$
S=.71
S=3/4

FIGURE 2

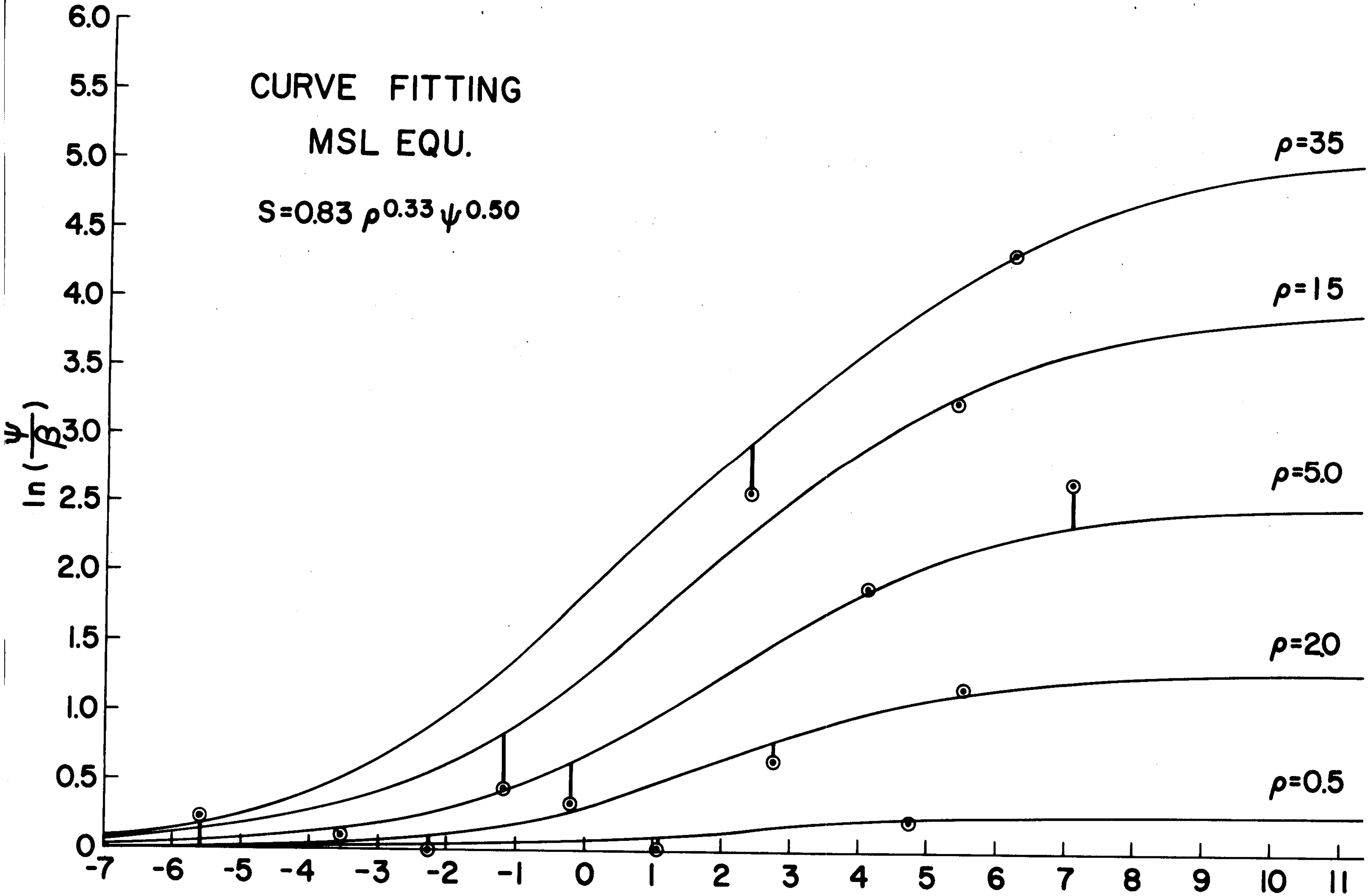

CURVE FITTING
MSL EQU.
S=0.83 ρ^0.33 ψ^0.50
ln(ψ/β)
ρ=35
ρ=15
ρ=5.0
ρ=2.0
ρ=0.5
ln ψ ⟶
6.0
5.5
5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11

FIGURE 3

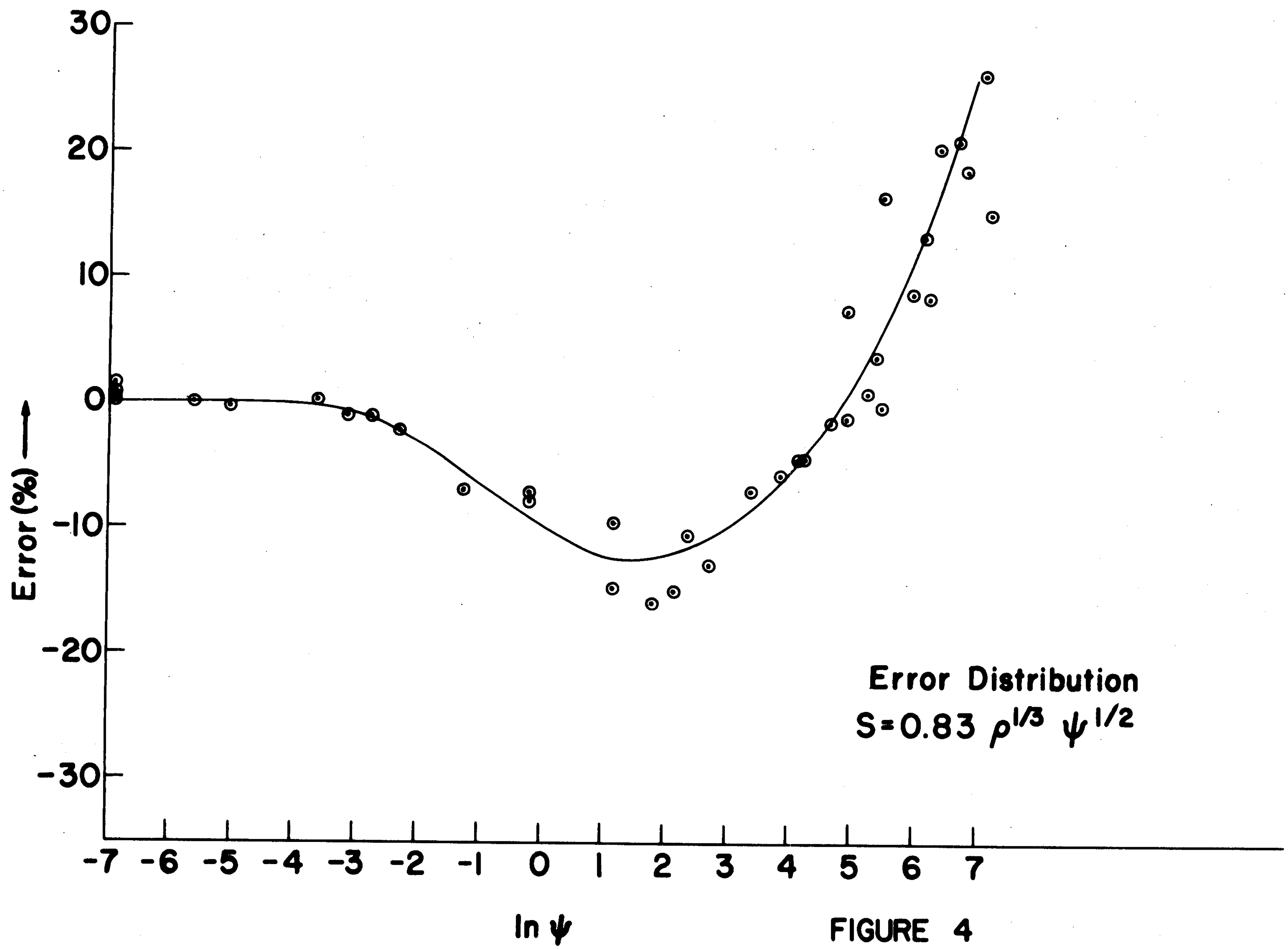

30
20
10
0
-10
-20
-30
Error (%)
-7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7
ln ψ
Error Distribution
S=0.83 ρ^1/3 ψ^1/2

FIGURE 4

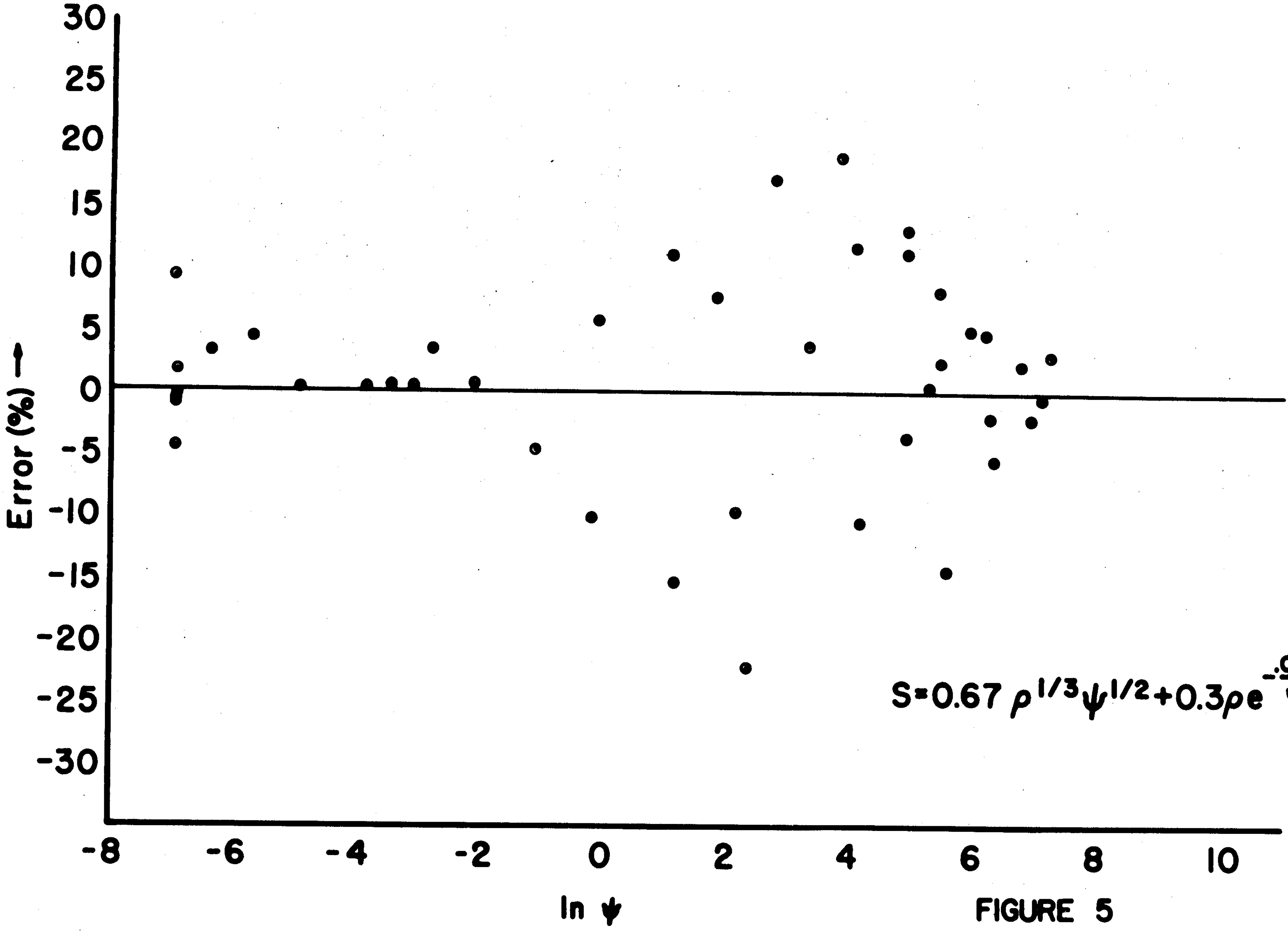

Error (%)
30
25
20
15
10
5
0
-5
-10
-15
-20
-25
-30
-8
-6
-4
-2
0
2
4
6
8
10
ln ψ
S=0.67 ρ^{1/3} ψ^{1/2} + 0.3 ρ e^{-.01/ψ}

FIGURE 5